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THE SOLUTION OF CONTACT PROBLEMS OF CREEP THEORY FOR COMBINED AGEING FOUNDATIONS*

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Solutions are presented of certain plane and axisymmetric contact problems on the frictionless impression of a rigid stamp into a two-layered ageing viscoelastic foundation. It is assumed that the upper layer is thin relative to the contact domain, and inhomogeneously ageing. The rheological properties of the lower layer are described by the equations of linear creep theory for ageing materials. The layers are mutually rigidly adherent. The contact domain does not change with time. Depending on the relationships between the moduli of the instantaneous elastic strains of the layers, the mixed problems reduce to integral equations of the first or second kinds containing Fredholm and Volterra operators. An analytic method is proposed for solving such equations which enables an expansion to be obtained for the fundamental characteristics of the contact interaction for a force varying with time in an arbitrary manner and acting on the stamp. Cases are investigaged for the artificial and natural ageing of a two-layer foundation.

1. We shall consider the problem using the example of axisymmetric problems, keeping in mind that it is possible to transfer to plane analogues of these problems by the correspondence principle /1/. Let a thin layer of thickness $0 \leqslant y \leqslant h$ $(ha^{-1} \ll 1)$ be rigidly adherent to a surface layer of thickness H lying frictionless on a non-deformable foundation (or connected to it). We assume that a force $P(t)$ is impressed without friction by a stamp of circular planform on the upper boundary of such a composite medium. The surface of the stamp foundation is given by the function $g(r)$, while the contact domain is determined by the inequality $0 \leqslant r \leqslant a$.

We will describe the rheological properties of the two-layer foundation by the equations of linear creep theory for ageing materials /2, 3/ (we ascribe the numbers $n = 1, 2$) to each layer from the top down)

$$e_{ij}^{(n)} = \frac{1+\nu_n}{E_n}\left[s_{ij}^{(n)} - \int_{\tau_0}^{t} s_{ij}^{(n)} K_n(t+\varkappa_n(z), \tau+\varkappa_n(z))\,d\tau\right]$$

$$\varepsilon^{(n)} = \frac{1-2\nu_n}{E_n}\left[\sigma^{(n)} - \int_{\tau_0}^{t} \sigma^{(n)} K_n(t+\varkappa_n(z), \tau+\varkappa_n(z))\,d\tau\right]$$

$$K_n(t,\tau) = E_n \frac{\partial}{\partial\tau} C_n(t,\tau)$$

Here $e_{ij}^{(n)}(t, r, z)$ and $s_{ij}^{(n)}(t, r, z)$ are strain and stress tensor deviators, $3\varepsilon^{(n)}(t, r, z)$ is the bulk strain, $\sigma^{(n)}(t, r, z)$ is the mean hydrostatic pressure, $K_n(t, \tau)$ is the creep kernel for the uniaxial state of stress, $C_n(t, \tau)$ is the creep measure, r and z are cylindrical coordinates of points of the body, τ_0 is the time of load application, $\varkappa_1(z)$ is a function of inhomogeneous ageing, $-\varkappa_2(z) = \tau_2$ is the time of lower layer fabrication, E_n and ν_n are the elastic instantaneous strain modulus and Poisson's ratio. We note that since the properties of the creep measure $C_n(t, \tau)$, as well as the creep kernel $K_n(t, \tau)$ and the relaxation $R_n(t, \tau)$ ($R_n(t, \tau)$ is the resolvent of the kernel $K_n(t, \tau)$) are elucidated in /2, 3/, we shall not duplicate them here. For simplicity in the subsequent considerations, we will merely assume that the hereditary properties of the layer materials are identical, i.e.,

$$C_n(t, \tau) = C_n(t-\tau, \tau) = \varphi_n(\tau) f(t-\tau) \quad (n = 1, 2) \tag{1.1}$$

where $\varphi_n(\tau)$ are functions taking account of the material ageing process and $f(t-\tau)$ their hereditary properties.

Now using the relative smallness of the thickness of the upper layer $(\Lambda = ha^{-1} \ll 1)$ we consider certain relationships between the values of the layer instantaneous elastic strain moduli.

Let

$$\theta_1\theta_2^{-1} \sim \Lambda^m \quad (\text{const} = m > 0;\ \Lambda \ll 1;\ \theta_n = 0{,}5E_n(1-\nu_n^2)^{-1};\ n = 1, 2).$$

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Then taking the results in /4, 5/ and the correspondence principle /2/ in the calculation we obtain an integral equation in the contact pressures $q(r, t)$ not known under the stamp. Inserting dimensionless variables and the notation

$$\rho^* = \rho a^{-1},\ r^* = ra^{-1},\ t^* = t\tau_0^{-1},\ \tau^* = \tau\tau_0^{-1} \tag{1.2}$$
$$\varkappa_1^*(z) = \varkappa_1(z)\tau_0^{-1},\ q^*(r^*, t^*) = q(r, t)\theta_2^{-1},\ \delta^*(t^*) = \delta(t)a^{-1}$$
$$g^*(r^*) = g(r)a^{-1},\ \lambda = Ha^{-1},\quad c = 0{,}5\Lambda\theta_2\theta_1^{-1}(1-2\nu_1)(1-\nu_1)^{-2}$$
$$C_n^*(t^*, \tau^*) = E_nC_n(t, \tau),\ P(t)(a^2\theta_2)^{-1} = N(t^*)$$

(we will omit the asterisk later), we write it in the form

$$c\left[q(r,t) - \int_1^t q(r,\tau)\bar{K}_1(t,\tau)\,d\tau\right] + \tag{1.3}$$
$$\int_0^1 \rho k\left(\frac{\rho}{\lambda}, \frac{r}{\lambda}\right)\left[q(\rho,t) - \int_1^t q(\rho,\tau)K_2(t-\tau_2, \tau-\tau_2)\,d\tau\right]d\rho =$$
$$\delta(t) - g(r)\quad (0 \leqslant r \leqslant 1,\ 1 \leqslant t \leqslant T < \infty)$$
$$\bar{K}_1(t,\tau) = \frac{1}{h}\int_0^h K_1(t + \varkappa_1(z), \tau + \varkappa_1(z))\,dz$$

$$k(\alpha, \beta) = \frac{1}{\lambda}\int_0^\infty L(u)J_0(u\alpha)J_0(u\beta)\,du \tag{1.4}$$

Here $\delta(t)$ is the rigid displacement of the stamp, $J_0(u)$ is the Bessel function, and the form of the function $L(u)$ is presented in /6/ for the cases of rigid clamping of the lower face of the second layer or its smooth contact with the non-deformable base.

The quasistatic condition

$$N(t) = 2\pi\int_0^1 \rho q(\rho, t)\,d\rho \tag{1.5}$$

must be appended to (1.3).

We now assume that $\Lambda\theta_1\theta_2^{-1} = D$ ($D = \text{const}$, $\Lambda \ll 1$), i.e., the instantaneous stiffness of the upper layer is greater than the instantaneous stiffness of the lower layer. In this case, a thin layer will work as an inhomogeneously-ageing cover /7/. If it is later assumed that the constant D is sufficiently small, then by changing to dimensionless variables and the notation (1.2), we obtain

$$\int_0^1 q(\rho, t)\rho k\left(\frac{\rho}{\lambda}, \frac{r}{\lambda}\right)d\rho - \int_1^t K_2(t-\tau_2, \tau-\tau_2)\,d\tau\ \times \tag{1.6}$$
$$\int_0^1 q(\rho, \tau)\rho k\left(\frac{\rho}{\lambda}, \frac{r}{\lambda}\right)d\rho = \delta(t) - g(r)$$
$$(0 \leqslant r \leqslant 1,\ 1 \leqslant t \leqslant T < \infty)$$

Therefore, we have arrived at a contact problem for a homogeneously-ageing viscoelastic layer.

2. We describe the method of solving integral equations (1.3) and (1.6) under the assumption that the force $N(t)$ pressing the stamp to the foundation varies with time as

$$N(t) = N_\infty + N_*(t)\ (N_\infty = \text{const};\ N_*(t) \to 0,\ t \to \infty) \tag{2.1}$$

We initially consider (1.3) and go over to its equivalent integral equation in conformity with the scheme elucidated in /8–10/:

$$Aq \equiv c\left[q(r,t) - q(r,1) - \int_1^t q(r,\tau)\bar{K}_1(t,\tau)\,d\tau\right] + \int_0^1 [q(\rho,t) - q(\rho,1)]\rho k\left(\frac{\rho}{\lambda}, \frac{r}{\lambda}\right)d\rho - \tag{2.2}$$
$$\int_1^t K_2(t-\tau_2, \tau-\tau_2)\,d\tau\int_0^1 q(\rho,\tau)\rho k\left(\frac{\rho}{\lambda}, \frac{r}{\lambda}\right)d\rho = \delta(t) - \delta(1)$$
$$(r \leqslant 1,\ 0 \leqslant t \leqslant T)$$

$$cq(r, 1) + \int_0^1 q(\rho, 1)\rho k\left(\frac{\rho}{\lambda}, \frac{r}{\lambda}\right)d\rho = \delta(1) - g(r)\quad (r \leqslant 1) \tag{2.3}$$

We seek the solution of (2.2) in the form /9—11/

$$q(r,t)=q_\infty(r)+q_*(r,t),\quad N_\infty=2\pi\int_0^1 \rho q_\infty(\rho)\,d\rho \tag{2.4}$$

$$\delta(t)=\delta+\delta_\infty\Delta(t)+\delta_*(t) \tag{2.5}$$

Then, taking account of (1.1) and (1.3) and the properties of the creep measure $C_n(t,\tau)$ /2, 3/, we obtain

$$\Delta(t)=\bar{C}_1(t,1),\ \Delta(1)=0,\ \bar{C}_1(t,\tau)=\bar{\varphi}_1(\tau)f(t-\tau) \tag{2.6}$$

$$\bar{\varphi}_1(\tau)=\frac{1}{h}\int_0^h \varphi_1(\tau+\varkappa_1(z))\,dz,\quad F=\frac{\varphi_2(1-\tau_2)}{\bar{\varphi}_1(1)}$$

$$cq_\infty(r)+F\int_0^1 q_\infty(\rho)\,\rho k\left(\frac{\rho}{\lambda},\frac{r}{\lambda}\right)d\rho=\delta_\infty\quad(r\leqslant 1) \tag{2.7}$$

$$Aq_*=\delta_*(t)-\delta_*(1)\ (r\leqslant 1,\ 0\leqslant t\leqslant T) \tag{2.8}$$

The relationship between the constants δ_∞ and N_∞ is found according to the second condition in (2.4) after solving integral equation (2.7) /1/.

We now require that

$$q_*(r,t)=q_1(t)+q_2(r,t) \tag{2.9}$$

$$\int_0^1 \rho q_2(\rho,t)\,d\rho=0,\quad 2\pi\int_0^1 \rho q_1(t)\,d\rho=\pi q_1(t)=N_*(t)$$

in conformity with the quasi-equilibrium condition (1.5) and the representation (2.1).

Now, if the expression

$$E\,[q_1(t)-q_1(1)],\ \int_0^1 [q_2(\rho,t)-q_2(\rho,1)]\,\rho B(\rho)\,d\rho$$

$$B(\rho)=2\int_0^1 \xi k\left(\frac{\rho}{\lambda},\frac{\xi}{\lambda}\right)d\xi,\quad E\int_1^t q_1(\tau)K_2(t-\tau_2,\tau-\tau_2)\,d\tau$$

$$\int_1^t K_2(t-\tau_2,\tau-\tau_2)\,d\tau\int_0^1 q_2(\rho,\tau)\,\rho B(\rho)\,d\rho$$

is added to and subtracted from the left side of (2.8) (E is as yet an undefined constant), then the integral equation (2.8) will be satisfied when the functions $q_1(t)$ and $q_2(r,t)$ will, respectively, yield the solutions of the equations

$$(c-E)[q_1(t)-q_1(1)]-c\int_1^t q_1(\tau)\bar{K}_1(t,\tau)\,d\tau=\delta_*(t)-\delta_*(1)- \tag{2.10}$$

$$\int_0^1 [q_2(\rho,t)-q_2(\rho,1)]\,\rho B(\rho)\,d\rho+E\int_1^t q_1(\tau)K_2(t-\tau_2,\tau-\tau_2)\,d\tau+$$

$$\int_1^t K_2(t-\tau_2,\tau-\tau_2)\,d\tau\int_0^1 q_2(\rho,\tau)\,\rho B(\rho)\,d\rho$$

$$c\left[q_2(r,t)-q_2(r,1)-\int_1^t q_2(r,\tau)\bar{K}_1(t,\tau)\,d\tau\right]+ \tag{2.11}$$

$$\int_0^1 [q_2(\rho,t)-q_2(\rho,1)]\,\rho k^\circ\left(\frac{\rho}{\lambda},\frac{r}{\lambda}\right)d\rho-$$

$$\int_1^t K_2(t-\tau_2,\tau-\tau_2)\,d\tau\int_0^1 q_2(\rho,\tau)\,\rho k^\circ\left(\frac{\rho}{\lambda},\frac{r}{\lambda}\right)d\rho=$$

$$h(r,t)\quad(r\leqslant 1,\quad 0\leqslant t\leqslant T)$$

$$h(r,t)=e(t)\,[E-{}^1\!/_2\,B(r)],\quad e(t)=q_1(t)-q_1(1)-\int_1^t q_1(\tau)K_2(t-\tau_2,\tau-\tau_2)\,d\tau$$

$$k^\circ(\rho/\lambda,\ r/\lambda)=k(\rho/\lambda,\ r/\lambda)-B(\rho)-B(r) \tag{2.12}$$

We note that the kernel $k^\circ(\alpha,\beta)$ of the form (2.12) is symmetric and possesses the property that

$$\int_0^1\int_0^1 q_2(\rho, t)\,\rho r k^\circ\left(\frac{\rho}{\lambda}, \frac{r}{\lambda}\right) d\rho\, dr = 0 \tag{2.13}$$

We introduce into the consideration the space $L_2^\circ(\Omega)$ of functions that are square integrable in the circle $\Omega: r \leqslant 1$ and satisfying the condition

$$\int_0^1 r h(r)\, dr = 0 \tag{2.14}$$

It can be proved that the space $L_2^\circ(\Omega)$ is a complete subspace of $L_2(\Omega)$.

Theorem 1. The integral operator

$$Hq = \int_0^1 q(\rho)\,\rho k^\circ\left(\frac{\rho}{\lambda}, \frac{r}{\lambda}\right) d\rho \tag{2.15}$$

is a selfadjoint, completely continuous and positive operator acting from $L_2^\circ(\Omega)$ into $L_2^\circ(\Omega)$.

Theorem 1 is proved by the scheme in /1/.

Furthermore, we construct a system of eigenfunctions $\{h_n(r)\}$ $(n \geqslant 1)$ and its corresponding sequence of eigennumbers $\{\mu_n\}$ of the operator (2.15) by the methods described in /1, 10/. By virtue of Theorem 1 this system is orthonormal and complete in $L_2^\circ(\Omega)$ and all $\mu_n \geqslant 0$, where $\mu_n \to 0$ $(n \to \infty)$.

Selecting the constant E in the second relationship in (2.11) in such a manner that a condition of the type (2.14) would be satisfied, i.e., $h(r, t) \in L_2^\circ(\Omega)$ in r, we represent the functions $q_2(r, t)$ and $h(r, t)$ in the form of the following series:

$$q_2(r, t) = \sum_{n=1}^{\infty} a_n(t)\, h_n(r) \tag{2.16}$$

$$h(r, t) = e(t) \sum_{n=1}^{\infty} c_n h_n(r), \qquad c_n = -\int_0^1\int_0^1 \rho r h_n(r)\, k\left(\frac{\rho}{\lambda}, \frac{r}{\lambda}\right) d\rho\, dr \tag{2.17}$$

Then inserting (2.16) and (2.17) into the integral equation (2.11), and equating coefficients of the left and right sides for eigenfunctions of identical number for the operator H in the relationship obtained, we arrive at the equation

$$a_n(t) - \int_1^t a_n(\tau)\, M_n(t, \tau)\, d\tau = a_n(1) + \frac{c_n}{c + \mu_n} e(t) \quad (1 \leqslant t \leqslant T)$$

$$M_n(t, \tau) = (c + \mu_n)^{-1} [c K_1(t, \tau) + \mu_n K_2(t - \tau_2, \tau - \tau_2)]$$

whose solution is representable in the form

$$a_n(t) = a_n(1)\left[1 + \int_1^t \Gamma_n(t, \tau)\, d\tau\right] + \frac{c_n}{c + \mu_n}\left[e(t) + \int_1^t e(\tau)\, \mathrm{T}_n(t, \tau)\, d\tau\right] \tag{2.18}$$

where $\Gamma_n(t, \tau)$ is the resolvent of the kernel $M_n(t, \tau)$ /2/.

Now, using formulas (2.9), (2.10), (2.16) and (2.18), we find the unknown addition, under the stamp, to the settling of the foundation $\delta_*(t)$ and the function $q_2(r, t)$ to an accuracy of a countable set of constants $a_n(1)$. We determine the latter by substituting the contact stresses $q(r, 1)$ into the integral equation (2.3) (the question of the solvability of the integral equation (2.3), just as of (2.7), is investigated in /1/) and by executing the following manipulation. We supplement the system $\{h_n(r)\}$ $(n \geqslant 1)$ of eigenfunctions of the operator Hq (2.15) by the element $h_0(r) = \sqrt{2}$. Then the sequence of functions $\{h_n(r)\}$ $(n \geqslant 0)$ will be orthonormal and complete in the space $L_2(\Omega)$. We expand the functions $g(r), B(r), q_\infty(r)$ belonging to $L_2(\Omega)$ in series in the system

$$g(r) = \sum_{n=0}^{\infty} g_n h_n(r), \quad B(r) = \sum_{n=0}^{\infty} b_n h_n(r) \tag{2.19}$$

$$q_\infty(r) = \sum_{n=0}^{\infty} d_n h_n(r)$$

Series (2.19) converge in the norm, at least, of the space $L_2(\Omega)$, and the corresponding coefficients belong to the space of quadratically summable sequences l_2 and are determined by using the orthonormalcy conditions for the functions $\{h_n(r)\}$ $(n \geqslant 0)$ We find from (2.1), (2.4), (2.9) and (2.16)

$$q(r, 1) = \sum_{n=0}^{\infty} X_n h_n(r), \quad X_0 = \frac{1}{\sqrt{2}\,\pi} N(1) \tag{2.20}$$

$$X_n = d_n + a_n(1) \quad (n \geqslant 1)$$

Taking account of the expansions (2.19) and (2.20), and the orthogonality condition for the system of functions $\{h_n(r)\}$ $(n \geqslant 0)$ we obtain from (2.3)

$$X_n = -(c+\mu_n)^{-1}(g_n + X_0 b_n/\sqrt{2}) \tag{2.21}$$

$$X_0 = (\sqrt{2}c + b_0)^{-1}[\delta(1) - \sum_{n=1}^{\infty} X_n b_n - \sqrt{2}g_0]$$

It was here also taken into account that $\{h_n(r)\}$ $(n \geqslant 1)$ are eigenfunctions of the operator (2.15) and satisfy condition (2.14). Now, we find the system of constants $a_n(1)$ from the first relationship of (2.12) and (2.20). We later determine $\delta_*(t)$ from (2.10) and the constant δ in (2.5) from the second equation in (2.21).

Theorem 2. The series (2.16) converges uniformly in $L_2^\circ(\Omega)$ in t in $[1, T]$, while (2.4), (2.9) and (2.16) determine the generalized solution of (1.3) and (1.4).

Without presenting the proof of Theorem 2, we note that it is analogous to that constructed in /11/.

3. We now consider the solution of integral equation (1.6). For this, as can be seen, it is necessary to set $\bar{K}_1(t,\tau) = c \equiv 0$ in the appropriate formulas in Sect.2. Then (2.3), (2.6), (2.7), (2.10) and (2.11) are rewritten in the form

$$\int_0^1 q(\rho,1)\rho k\left(\frac{\rho}{\lambda}, \frac{r}{\lambda}\right)d\rho = \delta(1) - g(r) \quad (r \leqslant 1) \tag{3.1}$$

$$\Delta(t) = \varphi_2(1-\tau_2)f(t-1), \quad \Delta(1) = 0$$

$$\int_0^1 q_\infty(\rho)\rho k\left(\frac{\rho}{\lambda}, \frac{r}{\lambda}\right)d\rho = \delta_\infty \quad (r \leqslant 1) \tag{3.2}$$

$$E\left[q_1(t) - q_1(1) - \int_1^t q_1(\tau)K_2(t-\tau_2, \tau-\tau_2)d\tau\right] = \delta_*(t) - \delta_*(1) + \int_1^t K_2(t-\tau_2,\tau-\tau_2)d\tau \int_0^1 q_2(\rho,\tau)\rho B(\rho)d\rho$$

$$\int_0^1 [q_2(\rho,t) - q_2(\rho,1)]\rho k^\circ\left(\frac{\rho}{\lambda}, \frac{r}{\lambda}\right)d\rho - \int_1^t K_2(t-\tau_2,\tau-\tau_2)d\tau \int_0^1 q_2(\rho,\tau)\rho k^\circ\left(\frac{\rho}{\lambda}, \frac{r}{\lambda}\right)d\rho = \tag{3.3}$$

$$h(r,t), \quad B(\rho) = \int_0^1 \frac{\xi}{\sqrt{1-\xi^2}} k\left(\frac{\rho}{\lambda}, \frac{\xi}{\lambda}\right)d\xi \quad (r \leqslant 1, \ 1 \leqslant t \leqslant T)$$

We note that the solutions of the integral equations (3.1)-(3.3) already have a singularity of square root type at the edge of the contact line $(r=1)$ /6/, unlike the preceding case, consequently, it is best to perform further discussion according to the following plan. Since the kernel $k^\circ(\alpha,\beta)$ (2.12) is symmetric and the equation

$$\int_0^1\int_0^1 \frac{\omega_2(\rho,t)\rho r}{\sqrt{(1-\rho^2)(1-r^2)}} k^\circ\left(\frac{\rho}{\lambda}, \frac{r}{\lambda}\right)d\rho\, dr = 0, \quad q_2(r,t) = \frac{\omega_2(r,t)}{\sqrt{1-r^2}}$$

is satisfied (compare with (2.13)), we introduce the space $\mathring{L}_{2,1/2}(\Omega)$ of functions summable square in the domain Ω and of weight $(1-r^2)^{-1/2}$ whose integral over Ω is zero. It can be shown that $\mathring{L}_{2,1/2}(\Omega)$ is a complete subspace of the Hilbert space $L_{2,1/2}(\Omega)$ that is square integrable with weight $(1-r^2)^{-1/2}$ in the domain Ω of functions. Moreover, as above we have /1/ the following theorem.

Theorem 3. The operator $H\omega$ (2.15), $q(r) = \omega(r)/\sqrt{1-r^2}$ is a selfadjoint, completely continuous, and positive-definite operator acting in the Hilbert space $\mathring{L}_{2,1/2}(\Omega)$.

As is shown in /10/, we now construct a system of eigenfunctions $\{h_n(r)\}$ $(n \geqslant 1)$ and its corresponding sequence of eigennumbers $\{\mu_n\}$ of the operator H by taking $\sqrt{4j+1}\,P_{2j}(\sqrt{1-r^2})$ ($P_j(r)$ are Legendre polynomials). Since by virtue of Theorem 3 the system $\{h_n(r)\}$ is orthonormal and complete in $\mathring{L}_{2,1/2}(\Omega)$ while $\mu_n > 0$, $\mu_n \to 0$ $(n \to \infty)$, then further solution of the problem reproduces the discussion in Sect.2 with appropriate evident modifications.

Furthermore, we turn our attention to one circumstance. To write down the integral equations of plane analogues of the problems studied, it is necessary to use the following correspondence principle /1, 6, 12/. If the kernel of a Fredholm integral equation of the axisymmetric contact problem represented in the form (1.4) is known, then the Fredholm kernel of the corresponding plane problem has the form

$$k(y) = \frac{1}{\pi}\int_0^\infty L(u)\,u^{-1}\cos uy\,du, \quad y = \frac{\xi - x}{\lambda} \tag{3.4}$$

The quasistatics conditions (1.5) is here transformed into the following:

$$N_0(t) = \int_{-1}^{1} q(x, t)\,dx, \quad N_1(t) = N_0(t)\,\varepsilon(t) = \int_{-1}^{1} xq(x, t)\,dx \tag{3.5}$$

which serve to determine the relationships between $N_0(t)$ and $\delta(t), \alpha(t)$ and $\varepsilon(t)$. Here $\varepsilon(t)$ and $\alpha(t)$ are, respectively, the eccentricity of application of the force $N_0(t)$, and the angle of rotation of the stamp.

As regards the solutions of the integral equations (1.3), (1.6), (3.4), (3.5), the methods elucidated in Sects.2 and 3 are applicable. It must just be kept in mind that in finding the eigenfunctions of the operator

$$Hq = \int_{-1}^{1} q(\xi)\,k\left(\frac{\xi - x}{\lambda}\right)d\xi$$

by the method in /10/, a system of orthonormal Legendre polynomials must be taken as coordinate elements in the first case, and Chebyshev polynomials of the first kind in the second.

4. We will analyse (1.3)–(1.5) numerically. Let

$$\varphi_1[\tau + \varkappa_1(z)] = A_0 + A_1\exp\{-\beta[\tau + \varkappa_1(z)]\}$$

Then in agreement with (2.6)

$$\bar{\varphi}_1(\tau) = A_0 + A_1\mu e^{-\beta\tau}, \quad \mu = \frac{1}{h}\int_0^h e^{-\beta\varkappa_1(z)}\,dz$$

We assume $\varkappa_1(z) \geqslant 0$, i.e., the age of the upper layer grows with height, as occurs if the layer is subjected to the influence of external effects (radiation, temperature, etc.), namely artificial ageing. In this case $0 < \mu \leqslant 1$. If $-1 < \varkappa_1(z) \leqslant 0$, i.e., the age of the thin layer diminishes with height which is natural ageing, and which corresponds to the process of raising the upper layer on the lower, then $1 \leqslant \mu < e^\beta$. Therefore, by changing the parameter μ within the mentioned limits, the solution of the problem posed can be constructed for any functions $\varkappa_1(z)$. Moreover, we note that the selection of the time origin can be made available so that $\tau_2 \equiv 0$.

Numerical computations were performed for the case when $g(r) = 0$ (the stamp has a flat base);

$$N(t) \equiv 1;\ \lambda = 6;\ c = 0.2;\ A_0 = 0.5522;\ A_1 = 4;\ f(t - \tau) = 1 - e^{-\gamma(t-\tau)};\ L(u) = (\text{ch}\,2u - 1)(\text{sh}\,2u + 2u)^{-1}$$

(values of the parameters β, γ and τ_0 were taken from /13/) and compared with those from /13/ in which the integral equation being investigaged (1.3)–(1.5) was solved by the methods described in /9–11/. Results of the comparison showed agreement between the numerical values of the fundamental characteristics of the problem under consideration with an error not exceeding 3%.

Let us recall certain mechanical deductions.

When $t = 1$ and for any value of the parameter μ the foundation will be an elastic layer bonded along the upper boundary by an elastic coating of Winkler type /5/. The minimal values of the contact pressures (for $r = 0$) will here be less than those in the case of natural ageing, and greater in the case of artificial ageing, while the maximum values (for $r = 1$) will be the reverse, less than the contact stresses for the case of artificial ageing, and greater in the case of natural ageing.

As the natural inhomogeneity grows (as the parameter u grows), the maximum contact pressures will diminish while the minimum pressures increase.

As the artificial inhomogeneity grows, which corresponds to a decrease in the parameter μ from 1 to 0, the maximum contact pressures will rise while the minimum pressures will decrease.

Settling of the foundation under the stamp $\delta(t)$ with time will grow and tend to the limit value which will be greater, the larger the parameter μ.

If the inhomogeneity parameter is $\mu = 1$ and the layers are fabricated from the very same material, while the force acting on the stamp from the flat foundation is independent of time, we obtain that the pressure distribution under the stamp will be the same as in the analogous elastic problem, i.e., in this case creep exerts no influence on the contact stress distribution.

Let $\mu = 0$. Then the upper layer will work on a type of foundation whose rheological properties are subject to the Volterra linear heredity law /14/.

The version $\mu = e^\beta$ corresponds to the case of piecewise-homogeneous ageing of the foundation under consideration and is investigated in /11/.

In conclusion, we note that a close connection exists between the contact problems for a composite viscoelastic foundations and the contact problems for linearly deformable foundations in the presence of abrasive wear /8, 15/. The same kind of integral equations as (1.3) or (1.6) occurs in investigation of the latter.

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